

ON THE SUP-NORM OF SL_3 HECKE-MAASS CUSP FORM

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ABSTRACT. This work contains a proof of a non-trivial bound in the eigenvalue aspect for the sup-norm of a $SL_3(\mathbb{Z})$ Hecke-Maass cusp form restricted to a compact set.

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1. INTRODUCTION

1.1. Statement of the results. The correspondence principle in quantum mechanics suggests a way to study a classical system via its semi-classical limit of quantization. For instance, let X be a compact Riemannian manifold. We can choose an orthonormal basis $(f_j)_{j \geq 0}$ of $L^2(X)$ satisfying

$$\forall j \geq 0, \quad \Delta(f_j) = \lambda_j f_j.$$

Date: Version of May 1, 2014.

1991 Mathematics Subject Classification. Primary 11F55, 11F60, 11F72, 11H55; Secondary 11D75, 43A90, 43A80.

Key words and phrases. Automorphic forms, sup-norm, pre-trace formula, amplification method, Paley-Wiener theorem, Helgason transform, spherical function.

where Δ is the Laplace-Beltrami operator on X and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ is its spectrum. If G^t is the geodesic flow on X then its quantization is $-\hbar^2 \Delta$, where \hbar is Planck's constant. Thus, it is very natural to attempt to understand the asymptotic behaviour of the eigenfunctions of Δ .

A classical question here – suggested by the correspondence principle – is to bound $\|f_j\|_\infty$ as $\lambda_j \rightarrow \infty$. (See [14] and [17] for more details.) A. Seeger and C. Sogge proved in [18] a very general and abstract bound, essentially sharp, in the case of compact Riemannian surfaces.

If X is a compact locally symmetric space then P. Sarnak proved in [16] the generic bound

$$\|f_j\|_\infty \ll \lambda_j^{(\dim(X) - \text{rank}(X))/4}$$

provided f_j is the joint eigenfunction of all the algebra of the invariant differential operators.

In [8], H. Iwaniec and P. Sarnak proved a bound sharper than that of A. Seeger and C. Sogge for arithmetic surfaces, which are the quotient of the upper-half plane by a congruence subgroup of $SL_2(\mathbb{Z})$ – both in the compact and in the non-compact case; they took advantage of the fact that some additional symmetries, the Hecke correspondences, act on these surfaces. The Laplace-Beltrami operator in this context is the hyperbolic Laplacian.

Following this foundational result, the sup-norm problem in the eigenvalue aspect has since been considered in various settings. For instance, S. Koyama investigated the case of quotients of the three-dimensional hyperbolic space by arithmetic subgroups in [10] and proved similar results, which have been recently improved by V. Blomer, G. Harcos and D. Milicevic in [2]. J. Vanderkam [20] and later on V. Blomer and P. Michel [3]) considered the case of the sphere and of the ellipsoids. S. Marshall considered the sup-norm problem restricted to totally geodesic submanifolds in [12] and in [13]. Very recently, V. Blomer and A. Pohl considered for the first time a manifold of higher rank and solved the case of Hecke Siegel Maass cusp form of genus 2 for $Sp_4(\mathbb{Z})$ in [1].

We will focus on another non-compact Riemannian symmetric space of dimension 5 and rank 2, which is

$$X = SL_3(\mathbb{Z}) \backslash SL_3(\mathbb{R}) / SO_3(\mathbb{R}).$$

In this manuscript, we provide a sketch of a non-trivial upper bound for a $SL_3(\mathbb{Z})$ Hecke-Maass cusp form at the point $z = I$ (the identity matrix) and at a generic point z in a fixed compact subset of X . Specifically, we establish the following result.

Theorem A – *Let Φ be a L^2 -normalized and tempered $SL_3(\mathbb{Z})$ Hecke-Maass cusp form on X with Laplace eigenvalue λ .*

- *We have*

$$\Phi(I) \ll_\varepsilon \lambda^{(5-2)/4 - 1/60 + \varepsilon}$$

for all $\varepsilon > 0$.

- *If C is a fixed compact in X then*

$$\|\Phi|_C\|_\infty \ll_{C,\varepsilon} \lambda^{(5-2)/4 - 1/124 + \varepsilon}$$

for all $\varepsilon > 0$.

A close inspection of the proof reveals that we get a better bound when the Langlands parameters of Φ get close to the walls of the dual positive Weyl chamber.

The method of proof builds on generalizations of the work of H. Iwaniec and P. Sarnak in [8], i.e. one studies a smooth amplified second moment, which comes from the spectral expansion

of an automorphic kernel, which itself has a geometric expansion. This is usually referred to the pre-trace formula.

A large amount of time is devoted to the construction of a relevant function on the spectral side of the pre-trace formula. In particular, one has to bound its inverse Helgason transform in the different domains of the positive Weyl chamber. This relies on the spherical inversion formula and on a systematic study of the $GL(3)$ spherical function itself.

Finally, the geometric side of the amplified pre-trace formula is bounded thanks to a counting lemma, which is the analogue of the one seen in [1].

1.2. Organization of the paper. Section 2 contains the knowledge on Lie groups and Lie algebras required for this work and all the relevant notations. Section 3 briefly explains the strategy of the proof and states an amplified pre-trace formula. The background on the $GL(3)$ Hecke algebra is given in Section 4. Moreover, several linearizations of compositions of some Hecke operators, which are required to make the amplification effective, are done. In Section 5, the function which occurs on the spectral side of the amplified pre-trace formula is constructed and several estimates for its inverse Helgason transform are proven. Section 6 contains a first bound for the geometric side of the amplified pre-trace formula, based on the results done in the previous sections. The counting lemma required to complete this bound is given in Section 7. The end of the proof of Theorem A appears in the final section.

Notations—The main parameters in this work are a positive real number T , which goes to infinity and a positive integer L (a power of T determined at the very final step) which goes to infinity with T . Thus, if f and g are some \mathbb{C} -valued functions on \mathbb{R}^2 then the symbols $f(T, L) \ll_A g(T, L)$ or equivalently $f(T, L) = O_A(g(T, L))$ mean that $|f(T, L)|$ is smaller than a constant, which only depends on A , times $g(T, L)$. Similarly, $f(T, L) = o(1)$ means that $f(T, L) \rightarrow 0$ as T goes to infinity among the positive real numbers.

We will denote by ε a positive constant whose value may vary from one line to the next one.

Acknowledgements—The authors would like to thank V. Blomer, F. Brumley, J. Cogdell, É. Fouvry, H. Iwaniec, E. Kowalski, E. Lapid, S. Marshall, P. Michel, A. Pohl, P. Sarnak and R. J. Stanton for stimulating exchange related to this project.

This paper was worked out at several places: while the three authors attended the workshop "Analytic theory of $GL(3)$ automorphic forms and applications" by the American Institute of Mathematics in Palo Alto, while the three authors were invited by Forschungsinstitut für Mathematik (FIM, ETH) in Zürich, while the first and second authors were invited by Université Blaise Pascal (Laboratoire de Mathématiques) in Clermont-Ferrand, while the second and third author were invited by the Ohio State University (Department of Mathematics) in Columbus. We would like to thank all these institutions for their hospitality and inspiring working conditions.

R. Holowinsky was supported by the Sloan fellowship BR2011-083 and the NSF grant DMS-1068043.

The research of G. Ricotta is supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme. The grant agreement number of this project, whose acronym is ANERAUTOHI, is PIEF-GA-2009-25271. He would like to thank ETH and its entire staff for the excellent working conditions.

2. BACKGROUND ON LIE GROUPS AND LIE ALGEBRAS

Let $G := SL_3(\mathbb{R})$ and

$$A = \left\{ a = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \in M_3(\mathbb{R}), \det(a) = 1, \forall i \in \{1, 2, 3\}, a_i > 0 \right\},$$

whose Lie algebra is

$$\mathfrak{a} = \left\{ H = \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix} \in M_3(\mathbb{R}), \text{Tr}(H) = 0 \right\},$$

whose complexification is denoted by $\mathfrak{a}_{\mathbb{C}}$. Let

$$N = \left\{ n = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \in M_3(\mathbb{R}) \right\}$$

and $K := SO_3(\mathbb{R})$ be one of the maximal compact subgroups of G .

The *Iwasawa decomposition* of G is given by $G = NAK$. If $g = nak$ then one denotes by

$$\text{Iw}_K(g) = k \text{ and } \text{Iw}_A(g) = a.$$

The set

$$\beta := \{H_{1,2} = E_{1,1} - E_{2,2}, H_{2,3} = E_{2,2} - E_{3,3}\}$$

is a basis of the 2-dimensional \mathbb{R} -vector space \mathfrak{a} . The *Killing form*

$$B(H, H') = \text{Tr}(HH')$$

is a positive definite quadratic form on \mathfrak{a} . The same properties hold for $\mathfrak{a}_{\mathbb{C}}$, the only difference being that the Killing form is a non-degenerate bilinear symmetric form on $\mathfrak{a}_{\mathbb{C}}$.

The \mathbb{R} -linear forms

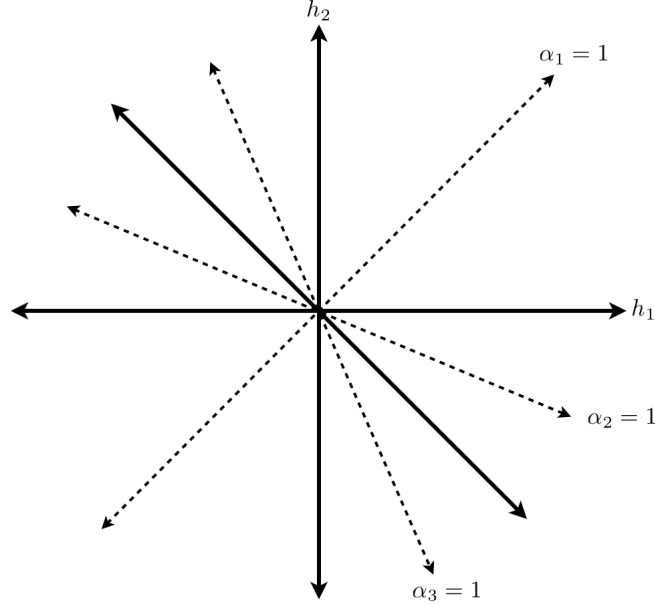
$$\alpha_{i,j}(H) = h_i - h_j$$

for $1 \leq i, j \leq 3$ belong to \mathfrak{a}^* and

$$\beta_1^* := \{\alpha_1^+ = \alpha_{1,2}, \alpha_2^+ = \alpha_{2,3}\}$$

is a basis of the 2-dimensional \mathbb{R} -vector space \mathfrak{a}^* , whose elements are called the *simple positive roots*. The last positive root is $\alpha_3^+ = \alpha_1^+ + \alpha_2^+$. The multiplicative roots on A are

$$\forall i \in \{1, 2, 3\}, \quad \alpha_i(a) = e^{\alpha_i^+(\log(a))}.$$

FIGURE 1. $A = \exp H$

Another basis of \mathfrak{a}^* is given by

$$\beta_2^* := \{\lambda_1, \lambda_2\}$$

where

$$\lambda_1(H) = h_1, \quad \lambda_2(H) = h_1 + h_2.$$

One can check that β_2^* is the dual basis of β . The same properties hold for $\mathfrak{a}_{\mathbb{C}}^*$.

The Killing form being positive definite on \mathfrak{a} , one can identify canonically \mathfrak{a} and \mathfrak{a}^* , in the sense that

$$\forall \lambda \in \mathfrak{a}^*, \exists ! H_\lambda \in \mathfrak{a}, \quad \lambda = B(H_\lambda, *).$$

In addition, one can transfer the Killing form to \mathfrak{a}^* by the formula

$$\forall (\lambda, \mu) \in (\mathfrak{a}^*)^2, \quad B(\lambda, \mu) := B(H_\lambda, H_\mu).$$

The basis β_2^* is the B -dual basis of the basis β_1^* , in the sense that

$$B(\lambda_i, \alpha_j^+) = \delta_{i,j}$$

for $1 \leq i, j \leq 2$. The same properties hold for $\mathfrak{a}_{\mathbb{C}}^*$ since the Killing form is non-degenerate on $\mathfrak{a}_{\mathbb{C}}^*$.

One can also define a positive definite quadratic form on $\mathfrak{a}_{\mathbb{C}}^*$ as follows. Obviously,

$$\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \oplus i\mathfrak{a}^*.$$

If $\lambda = \lambda_{\mathbb{R}} + \lambda_I$ with respect to this decomposition then the conjugate of λ is defined to be $\lambda^{\text{conj}} = \lambda_{\mathbb{R}} - \lambda_I$. The bilinear symmetric form on $\mathfrak{a}_{\mathbb{C}}^*$ given by

$$f(\lambda_1, \lambda_2) = B(\lambda_1, \lambda_2^{\text{conj}})$$

is positive definite and the induced norm is

$$\|\lambda\| = \sqrt{f(\lambda, \lambda)} = \sqrt{\|\lambda_{\mathbb{R}}\|^2 + \|\lambda_I\|^2}.$$

The basis β_2^* is the one that will be used to find an explicit integral representation for the spherical function. If λ belongs to $\mathfrak{a}_{\mathbb{C}}^*$ then there exists a unique pair $s = (s_1, s_2)$ of complex numbers satisfying

$$\lambda = s_1 \lambda_1 + s_2 \lambda_2.$$

One writes $\lambda = \lambda_s$.

One gets for free an explicit parametrization of the multiplicative characters on A via the exponential map. If a belongs to A then one can define

$$p_1(a) = a_1, \quad p_2(a) = a_1 a_2.$$

For $s = (s_1, s_2)$ a pair of complex numbers, the Selberg character of parameter s is given by

$$p_s(a) = p_1(a)^{s_1} p_2(a)^{s_2}.$$

A famous one is the module given by $\delta = p_{(1,1)}^2$. All the multiplicative characters of A are of this shape. If $\chi : A \rightarrow \mathbb{C}$ is a multiplicative character then there exists a unique pair $s = (s_1, s_2)$ of complex numbers satisfying

$$\chi = p_s.$$

Note that

$$\exp \circ \lambda_s = p_s \circ \exp.$$

The *Weyl group* W of G is the quotient of the normalizer of A in K by the centralizer of A in K . Its action on A can be identified to the action of the permutation group σ_3 by permuting the diagonal elements of the diagonal matrices in A . W also acts on \mathfrak{a} and $\mathfrak{a}_{\mathbb{C}}$ by permuting the diagonal matrices of these vector spaces. A fundamental domain for this action of W on A is given by the *positive Weyl chamber*

$$A_+ := \{a \in A, \alpha_1(a) > 1, \alpha_2(a) > 1\}.$$

This action is transferred by functoriality to an action on the group of multiplicative characters on A as follows. Recall that for $s \in \mathbb{C}^2$, the multiplicative character χ_s can be identified to the \mathbb{C} -linear function λ_s . For $w \in W$, one can define the \mathbb{C} -linear function $w.\lambda_s$ by

$$w.\lambda_s = B(w.H_{\lambda_s}, *)$$

In other words, $H_{w.\lambda_s} = w.H_{\lambda_s}$. The multiplicative character $w.p_s$ is the multiplicative character associated to $w.\lambda_s$, namely

$$(w.p_s)(a) = \exp((w.\lambda_s)(\log(a))).$$

Equivalently, W acts on \mathbb{C}^2 by the explicit formulas given by

$$\begin{aligned} (1, 2).s &= (-s_1, s_1 + s_2), \\ (1, 3).s &= (-s_2, -s_1), \\ (2, 3).s &= (s_1 + s_2, -s_2), \\ (1, 2, 3).s &= (s_2, -s_1 - s_2), \\ (1, 3, 2).s &= (-s_1 - s_2, s_1). \end{aligned}$$

Recall that the *Cartan decomposition* of G is $G = KAK$. If $g = k_1 a k_2$ then one has a simple formula for the *geodesic distance* on the Riemannian manifold G/K between g and the identity matrix I . Since I is fixed by the action of K , our distance function will only depend on the entries of a as

$$d(g, I)^2 := \log^2 a_1 + \log^2 a_2 + \log^2 a_3.$$

In terms of the multiplicative roots, this becomes

$$d(g, I)^2 = \frac{2}{3} (\log^2(\alpha_1(a)) + \log(\alpha_1(a)) \log(\alpha_2(a)) + \log^2(\alpha_2(a))).$$

3. THE AMPLIFIED PRE-TRACE FORMULA

Let Φ_{j_0} be our favorite $SL_3(\mathbb{Z})$ Hecke-Maass cusp form of type $\nu_0 = (\nu_{0,1}, \nu_{0,2}) \in \mathbb{C}^2$. The background on these objects can be found in [6]. One can include Φ_0 in an orthonormal basis of $SL_3(\mathbb{Z})$ Hecke-Maass cusp form $(\Phi_j)_{j \geq 0}$, the type of each Φ_j being $\nu_j = (\nu_{j,1}, \nu_{j,2}) \in \mathbb{C}^2$ for $j \geq 0$.

Let k be a smooth and compactly supported bi- K -invariant function on G satisfying the following properties.

- For $j \geq 0$, $\mathcal{H}(k)(\nu_j) \geq 0$ where $\mathcal{H}(k)$ is the Helgason transform of k (see Section 5).
- $\mathcal{H}(k)(\nu_{j_0}) \gg 1$.

Let $K(z, z')$ be the automorphic kernel given by

$$K(z, z') := \sum_{\gamma \in GL_3(\mathbb{Z})/\{\pm I\}} k(z^{-1}\gamma z')$$

for all z and z' in G . This function is left- $SL_3(\mathbb{Z})$ -invariant and right K -invariant with respect to each variable z and z' .

Spectrally decomposing via a pre-trace formula, one gets that

$$(3.1) \quad K(z, z') = \sum_{j \geq 0} \mathcal{H}(k)(\nu_j) \Phi_j(z') \overline{\Phi_j(z)} + \dots$$

where \dots stands for the contribution of the continuous spectrum.

Let I be a suitable finite subset of \mathbb{N}^2 and let $\alpha = (\alpha_{m,n})_{(m,n) \in I}$ be a suitable sequence of complex numbers. Defining

$$A_j(\alpha) := \sum_{(m,n) \in I} \alpha_{m,n} a_j(m, n)$$

one has that

$$\sum_{j \geq 0} |A_j(\alpha)|^2 \mathcal{H}(k)(\nu_j) \Phi_j(z') \overline{\Phi_j(z)} = \sum_{\substack{(m_1, n_1) \in I \\ (m_2, n_2) \in I}} \alpha_{m_1, n_1} \overline{\alpha_{m_2, n_2}} \sum_{j \geq 0} \mathcal{H}(k)(\nu_j) a_j(m_1, n_1) \overline{a_j(m_2, n_2)} \Phi_j(z') \overline{\Phi_j(z)}$$

upon expanding the square.

Assume the existence of linear operators $T_{m,n}$ and $T_{m,n}^*$ such that

$$\begin{aligned} T_{m,n}(\Phi_j) &= a_j(m, n) \Phi_j, \\ T_{m,n}^*(\Phi_j) &= \overline{a_j(m, n)} \Phi_j \end{aligned}$$

for $(m, n) \in I$. We shall later choose $a_j(m, n)$ to be the Hecke eigenvalues of certain Hecke operators $T_{m,n}$. Fix z and consider the previous equality as an equality of functions of z' . One has

$$\sum_{j \geq 0} |A_j(\alpha)|^2 \mathcal{H}(k)(v_j) \Phi_j(z') \overline{\Phi_j(z)} = \sum_{\substack{(m_1, n_1) \in I \\ (m_2, n_2) \in I}} \alpha_{m_1, n_1} \overline{\alpha_{m_2, n_2}} \sum_{j \geq 0} \mathcal{H}(k)(v_j) [(T_{m_2, n_2}^* \circ T_{m_1, n_1})(\Phi_j)](z') \overline{\Phi_j(z)}.$$

By (3.1), this gives

$$\sum_{j \geq 0} |A_j(\alpha)|^2 \mathcal{H}(k)(v_j) \Phi_j(z') \overline{\Phi_j(z)} = \sum_{\substack{(m_1, n_1) \in I \\ (m_2, n_2) \in I}} \alpha_{m_1, n_1} \overline{\alpha_{m_2, n_2}} [(T_{m_2, n_2}^* \circ T_{m_1, n_1})(K(z, *))](z').$$

The left-hand side of this formula is the *spectral side* whereas the right-hand side is the *geometric side* of the amplified pre-trace formula.

Choosing $z = z'$, one makes use of positivity of the summand and estimates the size of any single $\Phi_{j_0}(z)$ by the following inequality

$$(3.4) \quad |A_{j_0}(\alpha)|^2 \mathcal{H}(k)(v_{j_0}) |\Phi_{j_0}(z)|^2 \leq \sum_{\substack{(m_1, n_1) \in I \\ (m_2, n_2) \in I}} \alpha_{m_1, n_1} \overline{\alpha_{m_2, n_2}} [(T_{m_2, n_2}^* \circ T_{m_1, n_1})(K(z, *))](z).$$

Therefore, everything boils down to bounding the geometric side of the amplified pre-trace formula.

We will choose the coefficients $\alpha_{m,n}$ such that $|A_{j_0}(\alpha)|$ is bounded below by a small power of the main parameter T . We will also choose the coefficients $a_j(m, n)$ such that it will be possible to linearize the composition $T_{m_2, n_2}^* \circ T_{m_1, n_1}$. See Section 4 for an explicit description of all these parameters.

We will not choose the function k occurring in (3.4) but instead the function¹ $\mathcal{H}(k)$ with the required properties and we will prove the needed estimates for the corresponding function k in order to bound the geometric side of the amplified pre-trace formula.

4. THE HECKE ALGEBRA

4.1. Linearisations of Hecke operators. For g a matrix in $GL_3(\mathbb{Q})$, the Hecke operator T_g acts on a \mathbb{C} -valued function f defined on G , which is left- $SL_3(\mathbb{Z})$ -invariant and right- K -invariant, by the formula

$$(T_g(f))(z) = \sum_{\delta \in GL_3(\mathbb{Z}) \backslash GL_3(\mathbb{Z})gGL_3(\mathbb{Z})} f\left(\frac{1}{\det(\delta)^{1/3}} \delta z\right)$$

for all z in $SL_3(\mathbb{R})$. Note that on the one hand, the double coset $GL_3(\mathbb{Z})gGL_3(\mathbb{Z})$ is a finite union of left $GL_3(\mathbb{Z})$ cosets since g belongs to $GL_3(\mathbb{Q})$ and on the other hand, T_g is well-defined since its definition does not depend on a choice of representatives of the quotient set because f is left- $SL_3(\mathbb{Z})$ -invariant. The resulting new function $T_g(f)$ remains left- $SL_3(\mathbb{Z})$ -invariant and right- K -invariant. The fact that g is allowed to have rational coefficients and not only integer ones is required for the theory since the adjoint with respect to the Petersson inner product of T_g is $T_{g^{-1}}$.

¹Actually, similarly to what did H. Iwaniec and P. Sarnak in [8, Section 1], we will choose the inverse Fourier transform of $\mathcal{H}(k)$.

One can compute the action of such Hecke operator T_g on the automorphic kernel as follows. Let us fix a matrix z in G . One successively gets

$$\begin{aligned}
 (4.1) \quad (T_g(K(z, *))) (z') &= \sum_{\delta \in GL_3(\mathbb{Z}) \backslash GL_3(\mathbb{Z}) g GL_3(\mathbb{Z})} \sum_{\gamma \in GL_3(\mathbb{Z}) / \{\pm I\}} k \left(\frac{1}{\det(\delta)^{1/3}} z^{-1} \gamma \delta z' \right) \\
 &= \sum_{\delta \in GL_3(\mathbb{Z}) \backslash GL_3(\mathbb{Z}) g GL_3(\mathbb{Z})} \sum_{\gamma \in GL_3(\mathbb{Z}) / \{\pm I\}} k \left(\frac{1}{\det(\gamma \delta)^{1/3}} z^{-1} \gamma \delta z' \right) \\
 &= \sum_{\rho \in GL_3(\mathbb{Z}) g GL_3(\mathbb{Z}) / \{\pm I\}} k \left(\frac{1}{\det(\rho)^{1/3}} z^{-1} \rho z' \right)
 \end{aligned}$$

for all matrix z' in G . The equation (4.1) reveals that we should have a clear understanding of the double coset of g .

The main reference is [15]. Let $g = [g_{i,j}]_{1 \leq i, j \leq 3}$ be a matrix of size 3 with integer coefficients and $k \leq 3$ be a positive integer. Let I_k be the finite set of all k -tuples $\{i_1, \dots, i_k\}$ satisfying $1 \leq i_1 < \dots < i_k \leq 3$. If ω and τ are two elements of I_k then $g(\omega, \tau)$ will denote the $k \times k$ determinantal minor of g whose row indices are the elements of ω and whose column indices are the elements of τ . The k -th *determinantal divisor* of g say $d_k(g)$ is defined by

$$d_k(g) := \begin{cases} 0 & \text{if } \forall (\omega, \tau) \in I_k^2, g(\omega, \tau) = 0, \\ \gcd \{g(\omega, \tau), (\omega, \tau) \in I_k^2\} & \text{otherwise} \end{cases}$$

where the gcd is chosen to be positive. In particular,

$$d_1(g) = \gcd \{|g_{i,j}|, 1 \leq i, j \leq 3\}, \quad d_3(A) = |\det(g)|.$$

These quantities are useful since they completely determine a given double coset. More precisely, a matrix h of size 3 with integer coefficients belongs to $GL_3(\mathbb{Z})gGL_3(\mathbb{Z})$ if and only if

$$\forall 1 \leq k \leq 3, \quad d_k(h) = d_k(g).$$

The determinantal divisors satisfy the divisibility properties

$$\forall 1 \leq k \leq 2, \quad d_k(A)^2 \mid d_{k-1}(A) d_{k+1}(A)$$

with the convention $d_0(A) = 1$ and

$$d_1(A)^k \mid d_k(A)$$

for $1 \leq k \leq 3$.

For n a positive integer, the n -th normalized Hecke operator is defined by

$$T_n := \frac{1}{n} \sum_{\substack{g = \text{diag}(y_1, y_2, y_3) \\ y_1 \mid y_2 \mid y_3 \\ y_1 y_2 y_3 = n}} T_g.$$

Its dual with respect to the Petersson inner product is given by

$$T_n^* = \frac{1}{n} \sum_{\substack{g = \text{diag}(y_1, y_2, y_3) \\ y_1 \mid y_2 \mid y_3 \\ y_1 y_2 y_3 = n}} T_{g^{-1}}.$$

Applying the amplification method requires to being able to linearize the composition of several Hecke operators. The different required formulas are encapsulated in the following proposition.

Proposition 4.1—*Let p and q be two prime numbers.*

$$\begin{aligned} T_p \circ T_q &= \frac{1}{pq} T_{\text{diag}(1,1,pq)} + \delta_{p=q} \frac{p+1}{p^2} T_{\text{diag}(1,p,p)}, \\ T_p^* \circ T_q &= \frac{1}{pq} T_{\text{diag}(1,p,pq)} + \delta_{p=q} \frac{p^2+p+1}{p^2} \text{Id}, \\ T_p^* \circ T_q^* &= \frac{1}{pq} T_{\text{diag}(1,pq,pq)} + \delta_{p=q} \frac{p+1}{p^2} T_{\text{diag}(1,1,p)}. \end{aligned}$$

$$\begin{aligned} T_p \circ (T_q \circ T_q^* - \text{Id}) &= \frac{q+1}{pq^2} T_{\text{diag}(1,1,p)} + \frac{1}{pq^2} T_{\text{diag}(1,q,pq^2)} \\ &\quad + \delta_{p=q} \left(\frac{p+1}{p^3} T_{\text{diag}(1,p^2,p^2)} + \frac{p+1}{p^2} T_{\text{diag}(1,1,p)} \right). \end{aligned}$$

$$\begin{aligned} T_p^* \circ (T_q \circ T_q^* - \text{Id}) &= \frac{q+1}{pq^2} T_{\text{diag}(1,p,p)} + \frac{1}{pq^2} T_{\text{diag}(1,pq,pq^2)} \\ &\quad + \delta_{p=q} \left(\frac{p+1}{p^3} T_{\text{diag}(1,1,p^2)} + \frac{p+1}{p^2} T_{\text{diag}(1,p,p)} \right). \end{aligned}$$

$$\begin{aligned} (T_p \circ T_p^* - \text{Id}) \circ (T_q \circ T_q^* - \text{Id}) &= \frac{1}{p^2 q^2} T_{\text{diag}(1,pq,p^2 q^2)} + \frac{q+1}{p^2 q^2} T_{\text{diag}(1,p,p^2)} \\ &\quad + \frac{p+1}{p^2 q^2} T_{\text{diag}(1,q,q^2)} + \frac{(p+1)(q+1)}{p^2 q^2} \text{Id} \\ &\quad + \delta_{p=q} \left(\frac{p+1}{p^4} T_{\text{diag}(1,p^3,p^3)} + \frac{p+1}{p^4} T_{\text{diag}(1,1,p^3)} \right) \\ &\quad + \delta_{p=q} \left(\frac{(p+1)(2p-1)}{p^4} T_{\text{diag}(1,p,p^2)} + \frac{p(p+1)(1+p+p^2)}{p^4} \text{Id} \right). \end{aligned}$$

Recall that the Hecke algebra is isomorphic to the algebra of double $GL_3(\mathbb{Z})$ -cosets where the multiplication law is defined in [19]. The previous proposition follows from an explicit computation of the multiplication of the corresponding double cosets.

4.2. Constructing an amplifier. In this section, we will choose the set I and the coefficients $\alpha_{m,n}$, $(m,n) \in I$ occurring in (3.4).

Let us construct a relevant $GL(3)$ amplifier, based on the identity

$$(4.2) \quad a_{j_0}(1,p) a_{j_0}(p,1) - a_{j_0}(p,p) = 1$$

where $a_{j_0}(m,n)$ stands for the (m,n) -th Fourier coefficient of Φ_{j_0} . Let $L \geq 1$ be a parameter, whose value will be determined latter on (a positive power of T). Let us choose

$$I := \{(p,1), (1,p), (p,p), L \leq p \leq 2L, p \text{ prime}\}.$$

and

$$\alpha_{m,n} := \begin{cases} a_{j_0}(1, p) & \text{if } L \leq m = p \leq 2L \text{ is a prime and } n = 1, \\ a_{j_0}(p, 1) & \text{if } m = 1 \text{ and } L \leq n = p \leq 2L \text{ is a prime,} \\ -2 & \text{if } L \leq m = n = p \leq 2L \text{ are the same prime,} \\ 0 & \text{otherwise} \end{cases}$$

such that

$$\begin{aligned} A_{j_0}(\alpha) &= 2 \sum_{L \leq p \leq 2L} (a_{j_0}(1, p) a_{j_0}(p, 1) - a_{j_0}(p, p)) \\ &= 2 \sum_{L \leq p \leq 2L} 1 \\ &\gg_{\varepsilon} L^{1-\varepsilon} \end{aligned}$$

by (4.2). Moreover,

$$\begin{aligned} T_{p,1} = T_{1,p}^* &= T_p, \\ T_{p,1}^* = T_{1,p} &= T_p^*, \\ T_{p,p} = T_{p,p}^* &= T_p \circ T_p^* - \text{Id}. \end{aligned}$$

5. TEST FUNCTIONS IN THE PRE-TRACE FORMULA

5.1. Construction of a relevant test function on the spectral side. In this section, we will design the function $\mathcal{H}(k)$ occurring in (3.4).

If $F = \{a \in A, d(a, I) \geq 1\}$ then F is a closed subset of G , which does not contain I . K being compact, KFK and F share the same properties. Thus, one can find a Weyl-invariant symmetric open neighborhood O of I in G and a small enough positive real number δ satisfying

$$I \in O \subset A(\delta) = \{a \in A, ||\log a|| \leq \delta\} \subset G \setminus KFK$$

and $KA(\delta)K \subset G \setminus KFK = \{g \in G, d(g, I) < 1\}$.

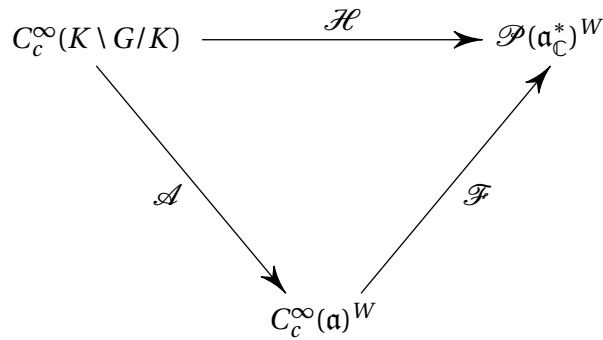


FIGURE 2. The Paley-Wiener theorem

The Paley-Wiener theorem asserts that the diagram given in figure 2 is a commutative diagram of isomorphisms of topological algebras. In this diagram, \mathcal{H} is the Helgason transform, \mathcal{F} is the Fourier transform and \mathcal{A} is the Abel transform. Of course, $C_c^\infty(\mathfrak{a})^W$ can be identified to $C_c^\infty(A)^W$,

via the exponential map. R. Gangolli proved a refined version in [5] of the Paley-Wiener theorem, which says that if g belongs to $C_c^\infty(A(\delta))^W$ then $\mathcal{A}^{-1}(g)$ belongs to $C_c^\infty(KA(\delta)A) \subset C_c^\infty(G \setminus KFK)$.

Both previous paragraphs imply that there exists a Weyl-invariant symmetric open neighborhood U of 0 in \mathfrak{a} such that

$$\forall g \in C_c^\infty(U)^W, \quad \mathcal{A}^{-1}(g) \in C_c^\infty(G \setminus KFK)$$

and $\|H\| \leq 1/3$ for H in U .

Let us fix U' a Weyl-invariant symmetric open neighborhood of 0 in \mathfrak{a} satisfying $U' + U' \subset U$. Let us also fix an even real non-negative function g in $C_c^\infty(U')^W$ normalised by $\int_{h \in \mathfrak{a}} g(h) dH = 2$. See figure 3.

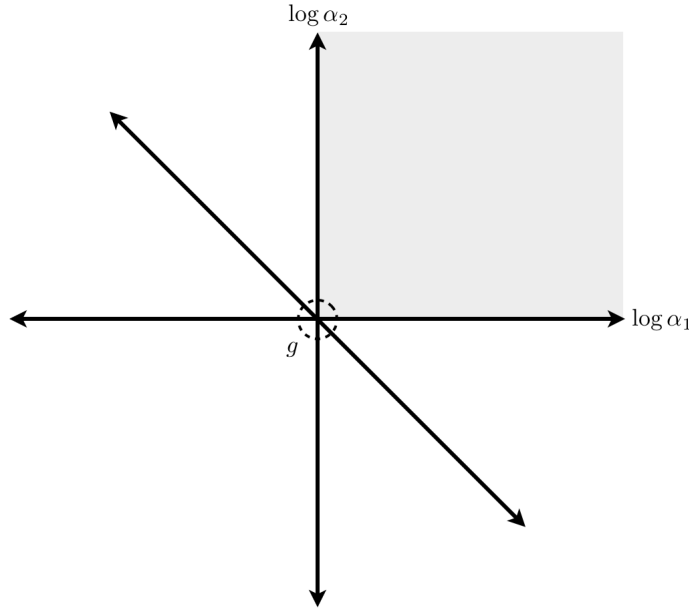


FIGURE 3. Test function $g \in C_c^\infty(\mathfrak{a})^W$

By [4, Lemma 6.2], the function $\mathcal{F}(g)$ in $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)^W$ is real-valued and satisfies

$$\forall \lambda \in \mathfrak{a}_\mathbb{C}^*, \quad \|\lambda\| \leq 1 \Rightarrow |\mathcal{F}(g)(\lambda)| \geq 1.$$

Recall that the Paley-Wiener condition means that

$$\forall \lambda \in \mathfrak{a}_\mathbb{C}^*, \forall m \geq 0, \quad |\mathcal{F}(g)(\lambda)| \leq c_m(g) \frac{\exp(\delta \|\lambda_\mathbb{R}\|)}{(1 + \|\lambda\|)^m}.$$

Briefly speaking, $\mathcal{F}(g)$ is a real bump function over 0.

In order to restore the positivity, let us define $h = g * g$ such that $\mathcal{F}(h) = \mathcal{F}(g)^2$. This function h is a real and even function in $C_c^\infty(U)^W$ such that $\mathcal{F}(h)$ is a non-negative function in $\mathcal{P}(\mathfrak{a}_\mathbb{C}^*)^W$ satisfying

$$\forall \lambda \in \mathfrak{a}_\mathbb{C}^*, \quad \|\lambda\| \leq 1 \Rightarrow \mathcal{F}(h)(\lambda) \geq 1.$$

The Paley-Wiener condition becomes

$$\forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \forall m \geq 0, \quad \mathcal{F}(h)(\lambda) \leq d_m(g) \frac{\exp(\delta^2 \|\lambda_{\mathbb{R}}\|)}{(1 + \|\lambda\|)^m}.$$

Thus, $\mathcal{F}(h)$ is a non-negative bump function over 0.

We would like to construct a bump function over the spectral parameter of our favorite tempered Hecke-Maass cusp form Φ_0 . The element λ_{Φ_0} of $\mathfrak{a}_{\mathbb{C}}^*$ associated to Φ_0 is given by

$$\lambda_{\Phi_0} = 3\nu_{0,1}\lambda_1 + 3\nu_{0,2}\lambda_2 \in i\mathfrak{a}^*$$

where $(\nu_{0,1}, \nu_{0,2})$ is the type of Φ_0 , which belongs to $i\mathbb{R}^2$ by the temperedness condition on Φ_0 . Let us define

$$\mu_T = 3iT\lambda_1 + 3iT\lambda_2$$

and

$$h_T = e^{-\mu_T} h \rightsquigarrow \mathcal{F}(h_T)(\lambda) = \mathcal{F}(h)(\lambda - \mu_T).$$

This function h_T belongs to $C_c^\infty(U)$ and its Fourier transform satisfies

$$\forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \quad \|\lambda - \mu_T\| \leq 1 \Rightarrow \mathcal{F}(h_T)(\lambda) \geq 1.$$

The Paley-Wiener condition becomes

$$(5.1) \quad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \forall m \geq 0, \quad \mathcal{F}(h_T)(\lambda) \leq d_m(g) \frac{\exp(\delta^2 \|\rho\|)}{(1 + \|\lambda - \mu_T\|)^m}.$$

This follows from the Paley-Wiener condition for h and the facts that $(\lambda - \mu_T)_{\mathbb{R}} = \lambda_{\mathbb{R}}$ with $\|\lambda_{\mathbb{R}}\| \leq \|\rho\|$ by [4, Proposition 3.4]. Thus, $\mathcal{F}(h_T)$ is a non-negative bump function over μ_T .

h_T not being Weyl-invariant, it seems natural to define

$$h_T^W(H) = \sum_{w \in W} h_T(w.h) = h(H) \sum_{w \in W} e^{-\mu_T(w.h)}$$

whose Fourier transform is given by

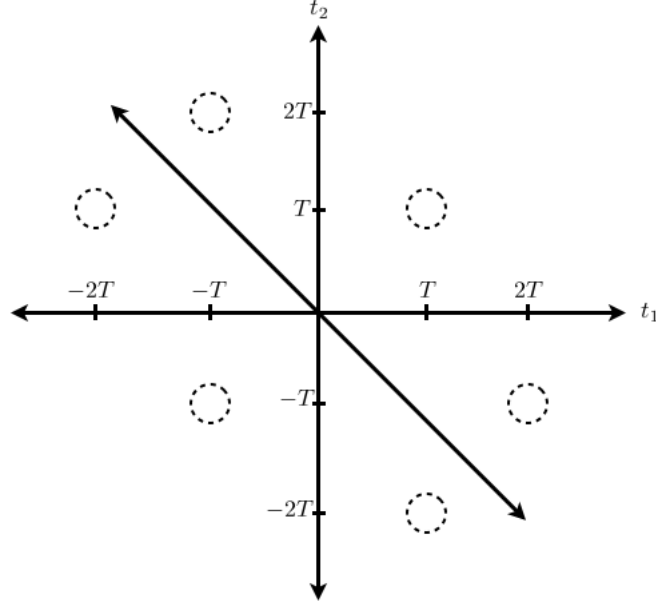
$$\mathcal{F}(h_T^W)(\lambda) = \sum_{w \in W} \mathcal{F}(h)(\lambda - w.\mu_T).$$

The previous paragraphs imply that h_T^W belongs to $C_c^\infty(U)^W$. The Fourier transform of h_T^W is non-negative and satisfies for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$$\mathcal{F}(h_T^W)(\lambda) \geq 1.$$

as soon as there exists w in W with $\|\lambda - w.\mu_T\| \leq 1$.

This function $\mathcal{F}(h_T^W)$ is the non-negative and Weyl-invariant bump function we were looking at (see figure 4). In other words, $\mathcal{H}(k) = \mathcal{F}(h_T^W)$ in (3.4).

FIGURE 4. Test function $\mathcal{F}(h_T^W) \in \mathcal{P}(\mathfrak{a}_{\mathbb{C}}^*)^W$

5.2. The spherical function.

5.2.1. *First formulas.* The *spherical function* of parameter $s \in \mathbb{C}^2$ is defined by

$$\varphi_s(g) = \int_{k \in K} (p_s \delta^{1/2})(\text{Iw}_A(kg)) dk$$

for g in G and where the Haar measure on K is normalized so that K has measure one. φ_s is a bi- K -invariant function on G , which is an eigenfunction of the algebra of invariant differential operators on G and normalized by $\varphi_s(I) = 1$.

The *third integral formula* for the spherical function (confer [9, Proposition 4.2(iii)]) asserts that

$$(5.2) \quad \varphi_s(a) = \kappa p_{s-1}(a) \int_{\bar{n} \in \bar{N}} p_{1-s}(\text{Iw}_A(a\bar{n}a^{-1})) p_{1+s}(\text{Iw}_A(\bar{n})) d\bar{n}$$

for $a \in A$. The constant κ is fixed by the condition $\varphi_s(I) = 1$.

In order to get a nice integral representation for the spherical function from (5.2), one needs to compute explicitly the Iwasawa projections on A of $a\bar{n}a^{-1}$ and of \bar{n} for a in A and \bar{n} in \bar{N} . The strategy suggested in [9, Chapter V Paragraph 5] leads to

$$\begin{aligned} \varphi_s(a) = \kappa p_{s-1}(a) \int_{(x,y,z) \in \mathbb{R}^3} & P(x,z)^{(-s_1-1)/2} Q(x,y,z)^{(-s_2-1)/2} \\ & \times P\left(\frac{x}{\alpha_1(a)}, \frac{z}{\alpha_3(a)}\right)^{(s_1-1)/2} Q\left(\frac{x}{\alpha_1(a)}, \frac{y}{\alpha_2(a)}, \frac{z}{\alpha_3(a)}\right)^{(s_2-1)/2} dx dy dz \end{aligned}$$

for $a \in A$ and where

$$\begin{aligned} P(X, Z) &= 1 + X^2 + Z^2, \\ Q(X, Y, Z) &= 1 + Y^2 + (XY - Z)^2. \end{aligned}$$

Note that

$$p_{s-1}(a) = \frac{1}{\alpha_1(a)\alpha_2(a)} \alpha_1(a)^{(2s_1+s_2)/3} \alpha_2(a)^{(s_1+2s_2)/3}.$$

5.2.2. *Integral representation on the walls of the positive Weyl chamber.* A particular role will be played by the walls of the positive Weyl chamber, i.e. the hyperplanes $\alpha_1(a) = 1$ and $\alpha_2(a) = 1$. Let us assume that a lies on the first wall $\alpha_1(a) = 1$. In that case, the change of variables

$$(x', y', z') = \left(x, \frac{y + xz}{1 + x^2}, z \right)$$

implies that

$$\begin{aligned} \varphi_s(a) &= \kappa p_{s-1}(a) \int_{(x,y,z) \in \mathbb{R}^3} P(x, z)^{(-s_1-1)/2} R(x, y, z)^{(-s_2-1)/2} \\ &\quad \times P\left(x, \frac{z}{\alpha_2(a)}\right)^{(s_1-1)/2} R\left(x, \frac{y}{\alpha_2(a)}, \frac{z}{\alpha_2(a)}\right)^{(s_2-1)/2} dx dy dz \end{aligned}$$

since

$$(1 + X^2)Q\left(X, \frac{Y + XZ}{1 + X^2}, Z\right) = 1 + X^2 + Y^2 + Z^2 = R(X, Y, Z).$$

The change of variables

$$(x', y', z') = \left(x, \frac{y}{\sqrt{1 + x^2}}, \frac{z}{\sqrt{1 + x^2}} \right)$$

implies that

$$\begin{aligned} \varphi_s(a) &= 4\pi\kappa p_{s-1}(a) \int_{(y,z) \in \mathbb{R}_+^2} (1 + z^2)^{(-s_1-1)/2} (1 + y^2 + z^2)^{(-s_2-1)/2} \\ &\quad \times \left(1 + \frac{z^2}{\alpha_2(a)^2}\right)^{(s_1-1)/2} \left(1 + \frac{y^2}{\alpha_2(a)^2} + \frac{z^2}{\alpha_2(a)^2}\right)^{(s_2-1)/2} dy dz. \end{aligned}$$

Finally, the change of variables

$$(U, V) = \left(\log\left(\alpha_2(a)^{2/3} \frac{1 + \frac{z^2}{\alpha_2(a)^2}}{1 + z^2}\right), \log\left(\alpha_2(a)^{4/3} \frac{1 + \frac{y^2}{\alpha_2(a)^2} + \frac{z^2}{\alpha_2(a)^2}}{1 + y^2 + z^2}\right) \right)$$

leads to

$$(5.3) \quad \varphi_s(a) = \pi\kappa \frac{\alpha_2(a)}{\sqrt{\alpha_2(a)^2 - 1}} \int_{U=-2L_2}^{L_2} \int_{V=-L_2}^{U+L_2} \theta(U, V) e^{\langle (s_1, s_2), (U, V) \rangle} dU dV$$

where

$$\theta(U, V) = \frac{1}{\sqrt{(e^{V+L_2} - 1)(e^{U+L_2-V} - 1)(e^{L_2-U} - 1)}}$$

and $L_2 = 2\log(\alpha_2(a))/3$.

The same formula holds on the other wall $\alpha_2(a) = 1$, up to the substitution of $\alpha_2(a)$ by $\alpha_1(a)$, after a different series of change of variables.

These integral representations for the $GL(3)$ spherical function at the neighborhood of the walls is related to the work done in [11].

5.2.3. *Asymptotic expansion in the positive Weyl chamber.* Let

$$\Delta(a) = (\alpha_1 \alpha_2)^2 (\alpha_1^2 - 1) (\alpha_2^2 - 1) (\alpha_3^2 - 1)$$

for $a \in A$. We would like to describe an asymptotic expansion for

$$\Delta(a)^{1/2} \varphi_s(a)$$

for s and a in specific domains.

Let us define the domain of validity for s of the next asymptotic expansion of the spherical function. Firstly,

$$\begin{aligned} \gamma_1 &= \{(0, s_2), s_2 \in \mathbb{C}\}, \\ \gamma_2 &= \{(s_1, 0), s_1 \in \mathbb{C}\}, \\ \gamma_3 &= \{(s_1, -s_1), s_1 \in \mathbb{C}\}. \end{aligned}$$

Then, for m_1, m_2 positive integers,

$$\sigma_{(m_1, m_2)} = \{(s_1, s_2) \in \mathbb{C}^2, m_1 s_1 + m_2 s_2 = m_1^2 + m_2^2 - 2m_1 m_2\}.$$

Finally, for m_1, m_2 positive integers and w_1, w_2 in W , let $\tau_{(m_1, m_2)}(w_1, w_2)$ be the subset of $s = (s_1, s_2) \in \mathbb{C}^2$ satisfying

$$(B(w_1.H_{\lambda_1}, H_1) - B(w_2.H_{\lambda_1}, H_1))s_1 + (B(w_1.H_{\lambda_2}, H_1) - B(w_2.H_{\lambda_2}, H_1))s_2 = 2m_1 - m_2$$

and

$$(B(w_1.H_{\lambda_1}, H_2) - B(w_2.H_{\lambda_1}, H_2))s_1 + (B(w_1.H_{\lambda_2}, H_2) - B(w_2.H_{\lambda_2}, H_2))s_2 = 2m_2 - m_1.$$

The parameter $s = (s_1, s_2)$ is said to be generic, say $s \in \mathbb{C}_{\text{gen}}^2$, if s does not belong to the union of hyperplanes given by

$$\bigcup_{j=1}^3 \gamma_j \cup \bigcup_{\substack{m_1, m_2 \geq 0 \\ m_1 m_2 \neq 0 \\ w \in W}} w \cdot \sigma_{(m_1, m_2)} \cup \bigcup_{\substack{m_1, m_2 \geq 0 \\ m_1 m_2 \neq 0 \\ w_1, w_2 \in W}} \tau_{(m_1, m_2)}(w_1, w_2)$$

R. Gangolli proved the following theorem in [5].

Theorem 5.1— *If s is generic and a belongs to A_+ then*

$$\Delta(a)^{1/2} \varphi_s(a) = p_{(-1, -1)}(a) \sum_{w \in W} c_3(w.s) p_{w.s}(a) \sum_{m_1, m_2 \geq 0} \frac{a_{m_1, m_2}(w.s)}{\alpha_1(a)^{m_1} \alpha_2(a)^{m_2}},$$

the series converging absolutely. If a belongs to a compact in A_+ then the previous series converges absolutely and uniformly. One has

$$a_{0,0} = 1.$$

Moreover, if $s = i(t_1, t_2) \in i\mathbb{R}^2$ then the coefficients involved in this series expansion are rational functions in (t_1, t_2) , which do not have poles on \mathbb{R}^2 and which satisfy

$$|a_{m_1, m_2}(s)| \leq C(m_1 + m_2)^{C'}$$

for all pair of non-negative integers $(m_1, m_2) \neq (0, 0)$ for some absolute constants C, C' and uniformly with respect to s and a .

Remark 5.2—Note that S. Helgason proved in [7] the Paley-Wiener theorem only for complex groups or real groups of rank 1. The previous theorem proved by R. Gangolli extends S. Helgason's result to the remaining cases.

5.3. Estimates for the inverse Helgason transform of our test function. The required estimates for the inverse Helgason transform k of our test function $\mathcal{F}(h_T^W)$ constructed in Section 5, which will enable us to estimate the geometric side of the amplified pre-trace formula (3.4) are given in the following proposition.

Proposition 5.3—Let a be an element in a compact subset of A .

- If a belongs to the closure of the positive Weyl chamber A_+ then

$$\mathcal{H}^{-1}(\mathcal{F}(h_T^W))(a) \ll T^3.$$

- If a belongs to the positive Weyl chamber A_+ then

$$\mathcal{H}^{-1}(\mathcal{F}(h_T^W))(a) \ll \frac{T^{3/2}}{\sqrt{(\alpha_1(a)^2 - 1)(\alpha_2(a)^2 - 1)(\alpha_3(a)^2 - 1)}}.$$

- If a satisfies $1 \leq \alpha_1(a) \leq 1 + O(1)/T$ and $\alpha_2(a) \geq 1 + O(1)/T$ then

$$\mathcal{H}^{-1}(\mathcal{F}(h_T^W))(a) \ll \frac{T^2}{\alpha_2(a)^2 - 1}.$$

- If a satisfies $\alpha_1(a) \geq 1 + O(1)/T$ and $1 \leq \alpha_2(a) \leq 1 + O(1)/T$ then

$$\mathcal{H}^{-1}(\mathcal{F}(h_T^W))(a) \ll \frac{T^2}{\alpha_1(a)^2 - 1}.$$

Altogether, the bounds given in this proposition are summarized in the figure 5.

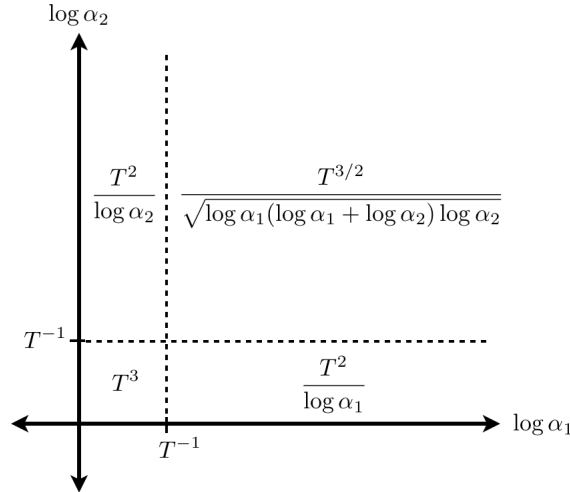


FIGURE 5. Bounds for the inverse Helgason transform $k = \mathcal{H}^{-1}(\mathcal{F}(h_T^W))$

Let us give a flavor of the proof for all these different bounds. The spherical inversion formula asserts that

$$(5.4) \quad \mathcal{H}^{-1}(\mathcal{F}(h_T^W))(a) = \int_{t \in \mathbb{R}^2} \mathcal{F}(h_T^W)(t) \varphi_{it}(a) \frac{dt}{|c_3(t)|^2},$$

the measure being the Plancherel one, where c_3 stands for the Harish-Chandra c-function. In particular, the Weyl-invariance of $\mathcal{F}(h_T^W)$, φ_{it} and of the Plancherel measure combined with the definition of h_T^W implies that

$$(5.5) \quad \mathcal{H}^{-1}(\mathcal{F}(h_T^W))(a) = 6 \int_{t \in \mathbb{R}^2} \mathcal{F}(h_T)(t) \varphi_{it}(a) \frac{dt}{|c_3(it)|^2}.$$

By (5.1), t is localized at (T, T) in (5.5), which implies that

$$|c_3(i(t_1, t_2))|^{-2} = t_1 t_2 (t_1 + t_2) \tanh(\pi t_1) \tanh(\pi t_2) \tanh(\pi(t_1 + t_2))$$

is of size T^3 , which is the first bound in Proposition 5.3 when a belongs to the closure of the positive Weyl chamber.

Let us assume that a belongs to the positive Weyl chamber. By Theorem 5.1, (5.4) and the Weyl invariance of $\mathcal{F}(h_T^W)$, the spherical function and of the Plancherel measure,

$$\mathcal{H}^{-1}(\mathcal{F}(h_T^W))(a) = 6 p_{(-1, -1)}(a) \Delta(a)^{-1/2} \int_{t \in \mathbb{R}^2} \mathcal{F}(h_T^W)(t) c_3(it) p_{it}(a) \sum_{m_1, m_2 \geq 0} \frac{a_{m_1, m_2}(it)}{\alpha_1(a)^{m_1} \alpha_2(a)^{m_2}} \frac{dt}{|c_3(it)|^2}.$$

Then,

$$\frac{c_3(it)}{|c_3(it)|^2} = \frac{1}{c_3(it)} = \frac{1}{c_3(-it)}$$

is bounded by $T^{3/2}$, which leads to the second bound in Proposition 5.3.

If a is close to the wall 1 defined by $\alpha_1(a) = 1$ and away from the wall 2 defined by $\alpha_2(a) = 1$, one can expand the function $\alpha_1(a) \mapsto \varphi_{it}(a)$ at 1 such that this is enough to investigate the case $\alpha_1(a) = 1$. By (5.5) and the integral representation (5.3) for the spherical function on the wall 1, one is lead to

$$(5.6) \quad 6\pi\kappa \frac{\alpha_2(a)}{\sqrt{\alpha_2(a)^2 - 1}} \int_{U=-2L_2}^{L_2} \int_{V=-L_2}^{U+L_2} \theta(U, V) D_3(h_T)(U, V) dU dV$$

by the Fourier inversion formula, where $L_2 = 2 \log(\alpha_2(a))/3$ and D_3 stands for the differential operator of order 3 given by

$$D_3 = \frac{\partial^3}{\partial x^2 y} + \frac{\partial^3}{\partial x y^2}.$$

Note that by the definition of h_T , if D_k is a differential operator of order $m \geq 0$ then

$$D_m(h_T) \ll T^m$$

such that one only recovers the bound T^3 from a trivial estimate in (5.6). The strategy is similar to what did H. Iwaniec and P. Sarnak in [8, Lemma 1.1], namely to integrate by parts in order to reduce the order of the differential order when the variables U and V are a little bit away from the polar lines $V = -L_2$, $U = L_2$ and $U - V = L_2$ and to estimate trivially in the remaining domains, whose volumes are quite small. Doing so, one can recover the third bound in Proposition 5.3.

6. FIRST ESTIMATE FOR THE GEOMETRIC SIDE OF THE AMPLIFIED PRE-TRACE FORMULA

The choice of the amplifier given in Section 4, the linearizations of the composition of Hecke operators given in Proposition 4.1, (3.4) and the properties of the function h_T^W constructed in the previous section imply that

$$\begin{aligned}
 (6.1) \quad L^{2-\varepsilon} |\Phi_{j_0}(z)|^2 &\ll_\varepsilon \sum_{L \leq p, q \leq 2L} \frac{|\alpha_{p,1} \alpha_{q,1}|}{pq} K_{q,pq}(z) + \sum_{L \leq p \leq 2L} \frac{|\alpha_{p,1}|^2 (p^2 + p + 1)}{p^2} K_{1,1}(z) \\
 &+ \sum_{L \leq p, q \leq 2L} \frac{|\alpha_{p,1} \alpha_{1,q}|}{pq} K_{1,pq}(z) + \sum_{L \leq p \leq 2L} \frac{|\alpha_{p,1}|^2 (p+1)}{p^2} K_{p,p}(z) \\
 &+ \sum_{L \leq p, q \leq 2L} \frac{|\alpha_{p,1}| (q+1)}{pq^2} K_{1,p}(z) + \sum_{L \leq p, q \leq 2L} \frac{|\alpha_{p,1}|}{pq^2} K_{q,pq^2}(z) \\
 &+ \sum_{L \leq p \leq 2L} \frac{|\alpha_{p,1}| (p+1)}{p^3} K_{p^2,p^2}(z) + \sum_{L \leq p \leq 2L} \frac{|\alpha_{p,1}| (p+1)}{p^2} K_{1,p}(z) \\
 &+ \sum_{L \leq p, q \leq 2L} \frac{1}{p^2 q^2} K_{pq,p^2 q^2}(z) + \sum_{L \leq p, q \leq 2L} \frac{q+1}{p^2 q^2} K_{p,p^2}(z) \\
 &+ \sum_{L \leq p, q \leq 2L} \frac{p+1}{p^2 q^2} K_{q,q^2}(z) + \sum_{L \leq p, q \leq 2L} \frac{(p+1)(q+1)}{p^2 q^2} K_{1,1}(z) \\
 &+ \sum_{L \leq p \leq 2L} \frac{p+1}{p^4} K_{p^3,p^3}(z) + \sum_{L \leq p \leq 2L} \frac{p+1}{p^4} K_{1,p^3}(z) \\
 &+ \sum_{L \leq p \leq 2L} \frac{(p+1)(2p-1)}{p^4} K_{p,p^2}(z) + \sum_{L \leq p \leq 2L} \frac{p(p+1)(1+p+p^2)}{p^4} K_{1,1}(z)
 \end{aligned}$$

where

$$K_{\ell,n}(z) := \sum_{\rho \in GL_3(\mathbb{Z}) \text{diag}(1,\ell,n) GL_3(\mathbb{Z}) / \{\pm 1\}} \left| k \left(\frac{1}{\det(\rho)^{1/3}} z^{-1} \rho z \right) \right|.$$

The quantities $K_{\ell,n}(z)$ will be bounded thanks to Proposition 5.3 and a counting lemma given in the next section.

7. COUNTING LEMMAS

In this section, ρ will denote a matrix of size 3 with integer coefficients satisfying

$$(d_1(\rho), d_2(\rho), d_3(\rho)) = (1, \ell, n)$$

with $\ell \mid n$ and z will be a point in a compact subset of X , which means that

$$z = naK = \begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{pmatrix} \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} K$$

where x_1, x_2 and x_3 belong to an interval of length 1 and

$$1 \gg \beta_1 = \beta_1(a) := \frac{a_1}{a_2}, \beta_2 = \beta_2(a) := \frac{a_2}{a_3} \geq \frac{\sqrt{3}}{2}.$$

The Cartan decomposition of $z^{-1}\rho z$ can be written as

$$z^{-1}\rho z = k_1 b k_2.$$

By a slight abuse of notations, one can set

$$\alpha_1 = \alpha_1(z^{-1}\rho z) := \alpha_1(b), \quad \alpha_2 = \alpha_2(z^{-1}\rho z) := \alpha_2(b).$$

Let $M_{\ell,n}(z; \delta_1, \delta_2)$ be the number of such matrices ρ satisfying

$$1 \leq \alpha_1 \leq 1 + \delta_1, \quad 1 \leq \alpha_2 \leq 1 + \delta_2$$

where $0 \leq \delta_1, \delta_2 \ll 1$.

Proposition 7.1—Let $\Delta = \delta_1^2 + \delta_2^2 + \delta_1\delta_2$, n a positive integer, which goes to infinity with T and ℓ a positive integer dividing n .

- At the identity matrix I ,

$$M_{\ell,n}(I; \delta_1, \delta_2) \ll_{\varepsilon} n^{1/3+\varepsilon} \sum_{\lambda|\ell} \frac{1}{\lambda} \left(1 + n^{2/3}(\sqrt{\Delta} + \Delta)\right)^2 \left(1 + \frac{n^{2/3}(\sqrt{\Delta} + \Delta)}{\ell/\lambda}\right) \left(1 + \frac{n^{1/3}(\sqrt{\Delta} + \Delta)}{\ell/\lambda}\right).$$

- If z belongs to a compact subset of X then

$$M_{\ell,n}(z; \delta_1, \delta_2) \ll_{\varepsilon} n^{1/3+\varepsilon} \sum_{\lambda|\ell} \frac{1}{\lambda} \left(1 + n^{2/3}(\sqrt{\Delta} + \Delta)^{1/5}\right)^2 \left(1 + \frac{n^{2/3}(\sqrt{\Delta} + \Delta)^{1/5}}{\ell/\lambda}\right) \left(1 + \frac{n^{1/3}(\sqrt{\Delta} + \Delta)}{\ell/\lambda}\right).$$

This counting lemma is optimal in the following sense. If $z = I$ then the number of matrices ρ in K is bounded by $n^{1/3+\varepsilon}$ if n is a cube, which matches the order of magnitude for the number of automorphs of I , namely the number of matrices ρ satisfying $\rho K = K$.

Let us give a flavor of the proof of this counting lemma at I . Let

$$\rho = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

be one of the matrices we would like to count and let assume for simplicity that $g \neq 0$ is co-prime with ℓ and that $(dj - fg)(dh - eg) \neq 0$.

The first usual step is to prove that

$$(7.1) \quad \rho = n^{1/3} k_1 k_2 + O\left(n^{1/3} \sqrt{\Delta}\right).$$

The matrix $k_1 k_2$ being orthogonal, (7.1) implies that

$$(7.2) \quad q(a, d) = n^{2/3} - g^2 + O\left(n^{2/3}(\sqrt{\Delta} + \Delta)\right),$$

$$(7.3) \quad q(h, j) = n^{2/3} - g^2 + O\left(n^{2/3}(\sqrt{\Delta} + \Delta)\right),$$

$$(7.4) \quad q(e, f) = n^{2/3} - d^2 + O\left(n^{2/3}(\sqrt{\Delta} + \Delta)\right),$$

$$(7.5) \quad b = \frac{-1}{n^{1/3}}(fg - dj) + O\left(n^{1/3}(\sqrt{\Delta} + \Delta)\right)$$

where $q(x, y) = x^2 + y^2$ is obviously positive definite.

There are $n^{1/3}$ coefficients g by (7.1). Let us fix one of them.

There are $1 + n^{2/3}(\sqrt{\Delta} + \Delta)$ pairs of coefficients (a, d) by (7.2) and $1 + n^{2/3}(\sqrt{\Delta} + \Delta)$ pairs of coefficients (h, j) by (7.3). Let us fix one of these quadruples (a, d, h, j) .

Note that e and f belong to some fixed congruence classes modulo ℓ since ℓ divides both $dj - fg$ and $dh - eg$. By (7.4), there are $1 + n^{2/3}(\sqrt{\Delta} + \Delta)/\ell$ pairs of coefficients (e, f) . Let us fix one of these pairs (e, f) .

b belongs to a fixed congruence class modulo ℓ since ℓ divides $ah - gb$. Thus, there are $1 + n^{1/3}(\sqrt{\Delta} + \Delta)/\ell$ coefficients b by (7.5). Let us fix one of these coefficients b .

c is fixed by the determinant equation.

Let us mention that the proof at a generic point z in a compact subset of X follows the same lines, the only difference being that the different quadratic forms encountered are still positive definite but their coefficients are real numbers, which depend on x_1, x_2, x_3 and on the multiplicative roots β_1, β_2 . Actually, the discriminants of these quadratic forms will be either $\beta_1 \geq \sqrt{3}/2 > 0$ or $\beta_2 \geq \sqrt{3}/2 > 0$, which enable us to approximate them by positive definite quadratic forms with rational coefficients. This Diophantine approximation preliminary step lies at the heart of the proof of the counting lemma proved by V. Blomer and A. Pohl ([1]).

8. END OF THE PROOF OF THEOREM A AT I

The following proposition gives a bound for the quantities $K_{\ell,n}(z)$. Let us define

$$M_{\ell,n} := \sum_{\lambda|\ell} \frac{1}{\lambda} \left(1 + \frac{n^{2/3}}{\ell/\lambda}\right) \left(1 + \frac{n^{1/3}}{\ell/\lambda}\right).$$

Proposition 8.1—Let $\Delta = \delta_1^2 + \delta_2^2 + \delta_1\delta_2$, n a positive integer, which goes to infinity with T and ℓ a positive integer dividing n .

- At the identity matrix I , if $n \leq T^{3/2}$ then

$$K_{\ell,n}(I) \ll_{\varepsilon} T^3(\ell n)^{1/3+\varepsilon} + T^2(\ell n)^{7/3+\varepsilon} M_{\ell,\ell n} + T^{3/2}(\ell n)^{8/3+\varepsilon} M_{\ell,\ell n}.$$

- If z belongs to a compact subset of X and $n \leq T^{3/10}$ then

$$K_{\ell,n}(z) \ll_{\varepsilon} T^3(\ell n)^{1/3+\varepsilon} + T^2(\ell n)^{5+\varepsilon} M_{\ell,\ell n} + T^{3/2}(\ell n)^{20/3+\varepsilon} M_{\ell,\ell n}.$$

Let us define

$$\alpha := \begin{cases} 1 & \text{if } z = I, \\ 1/5 & \text{if } z \text{ belongs to a compact subset of } X. \end{cases}$$

This proposition follows trivially from the bounds proved in Proposition 5.3 for the inverse Helgason transform of h_T^W and from Proposition 7.1. More precisely, Proposition 5.3 implies that

$$\mathcal{H}^{-1}(h_T^W)(a) \ll \begin{cases} T^3 & \text{if } 1 \leq \alpha_1(a), \alpha_2(a) \leq 1 + 1/n^{2/3\alpha}, \\ T^2 n^{2/3\alpha} & \text{if } 1 + 1/n^{2/3\alpha} \leq \alpha_1(a) \ll 1, 1 \leq \alpha_2(a) \leq 1 + 1/n^{2/3\alpha}, \\ T^2 n^{2/3\alpha} & \text{if } 1 \leq \alpha_1(a) \leq 1 + 1/n^{2/3\alpha}, 1 + 1/n^{2/3\alpha} \leq \alpha_2(a) \ll 1, \\ T^{3/2} n^{1/\alpha} & \text{if } 1 + 1/n^{2/3\alpha} \leq \alpha_1(a), \alpha_2(a) \ll 1 \end{cases}$$

and Proposition 7.1 entails that

$$M_{\ell,n}(z; \delta_1, \delta_2) \ll \begin{cases} n^{1/3+\varepsilon} & \text{if } 0 \leq \delta_1, \delta_2 \leq 1/n^{2/3\alpha}, \\ n^{5/3+\varepsilon} M_{\ell,n} & \text{if } 1/n^{2/3\alpha} \leq \delta_1 \ll 1, 0 \leq \delta_2 \leq 1/n^{2/3\alpha}, \\ n^{5/3+\varepsilon} M_{\ell,n} & \text{if } 0 \leq \delta_1 \leq 1/n^{2/3\alpha}, 1/n^{2/3\alpha} \leq \delta_2 \ll 1, \\ n^{5/3+\varepsilon} M_{\ell,n} & \text{if } 1/n^{2/3\alpha} \leq \delta_1, \delta_2 \ll 1. \end{cases}$$

Let us quickly finish the proof of Theorem A at $z = I$. By (6.1), Proposition 8.1 and Rankin-Selberg theory, one gets

$$|\Phi_{j_0}(I)|^2 \ll_{\varepsilon} L^{\varepsilon} \left(\frac{T^3}{L} + T^2 L^{14} + T^{3/2} L^{16} \right).$$

The optimal choice for L is given by $L = T^{1/15}$, which implies Theorem A at $z = I$.

REFERENCES

- [1] V. Blomer and A. Pohl, *The sup-norm problem on the siegel modular space of rank two*. preprint available at <http://arxiv.org/abs/1402.4635>.
- [2] V. Blomer, G. Harcos, and D. Milicevic, *Eigenfunctions on arithmetic hyperbolic 3-manifolds*. available at <http://arxiv.org/abs/1401.5154>.
- [3] V. Blomer and P. Michel, *Sup-norms of eigenfunctions on arithmetic ellipsoids*, Int. Math. Res. Not. IMRN **21** (2011), 4934–4966. MR2852302
- [4] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, *Spectra of compact locally symmetric manifolds of negative curvature*, Invent. Math. **52** (1979), no. 1, 27–93. MR532745 (82a:58050a)
- [5] R. Gangolli, *On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. of Math. (2) **93** (1971), 150–165. MR0289724 (44 #6912)
- [6] D. Goldfeld, *Automorphic forms and L-functions for the group $GL(n, \mathbb{R})$* , Cambridge Studies in Advanced Mathematics, vol. 99, Cambridge University Press, Cambridge, 2006. With an appendix by Kevin A. Broughan. MR2254662 (2008d:11046)
- [7] S. Helgason, *An analogue of the Paley-Wiener theorem for the Fourier transform on certain symmetric spaces*, Math. Ann. **165** (1966), 297–308. MR0223497 (36 #6545)
- [8] H. Iwaniec and P. Sarnak, *L^{∞} norms of eigenfunctions of arithmetic surfaces*, Ann. of Math. (2) **141** (1995), no. 2, 301–320. MR1324136 (96d:11060)
- [9] J. Jorgenson and S. Lang, *Spherical inversion on $SL_n(\mathbb{R})$* , Springer Monographs in Mathematics, Springer-Verlag, New York, 2001. MR1834111 (2002j:22013)
- [10] S.-Y. Koyama, *L^{∞} -norms on eigenfunctions for arithmetic hyperbolic 3-manifolds*, Duke Math. J. **77** (1995), no. 3, 799–817. MR1324641 (96d:11057)
- [11] B. Krötz and R. J. Stanton, *Holomorphic extensions of representations. I. Automorphic functions*, Ann. of Math. (2) **159** (2004), no. 2, 641–724. MR2081437 (2005f:22018)
- [12] S. Marshall, *Geodesic restrictions of arithmetic eigenfunctions*. available at <http://www.math.northwestern.edu/~slm/research.html>.
- [13] ———, *Restrictions of SL_3 maass forms to maximal flat subspaces*. available at <http://www.math.northwestern.edu/~slm/research.html>.
- [14] N. Nadirashvili, Dzh. Tot, and D. Yakobson, *Geometric properties of eigenfunctions*, Uspekhi Mat. Nauk **56** (2001), no. 6(342), 67–88. MR1886720 (2002k:35228)
- [15] M. Newman, *Integral matrices*, Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 45. MR0340283 (49 #5038)
- [16] P. Sarnak, *Letter to Morawetz*. available at <http://www.math.princeton.edu/sarnak>.
- [17] ———, *Arithmetic quantum chaos*, The Schur lectures (1992) (Tel Aviv), 1995, pp. 183–236. MR1321639 (96d:11059)
- [18] A. Seeger and C. D. Sogge, *Bounds for eigenfunctions of differential operators*, Indiana Univ. Math. J. **38** (1989), no. 3, 669–682. MR1017329 (91f:58097)

- [19] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1. MR1291394 (95e:11048)
- [20] J. M. VanderKam, L^∞ norms and quantum ergodicity on the sphere, *Internat. Math. Res. Notices* **7** (1997), 329–347. MR1440572 (99d:58175)

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